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A TYPE OF HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION.

BY L. A. HOWLAND.

1. The object of this paper is to discuss the homogeneous linear differential equation

$$(1) \qquad A(y) = y^{(n)} + \sum_{k=0}^{n-1} a_k y^{(k)} = 0$$

(the a's being analytic functions of x in a region R and not all constant*) *which has a fundamental system of integrals of the form*

$$(2) \qquad y_1, y_1', y_1'', \cdots y_1^{(n-1)}.$$

A necessary condition for this is that there be no relation of the form

$$\sum_{h=0}^{n-1} c_h y_1^{(h)} \equiv 0,$$

where the c's are constants, not all zero; that is, y_1 may not be an integral of an equation of order less than n with constant coefficients.

A further necessary condition is obtained as follows: the equation

$$A'(y) = y^{(n+1)} + \sum_{k=0}^{n-1} a_k y^{(k+1)} + \sum_{k=0}^{n-1} a_k' y^{(k)} = 0$$

has the integrals (2) and, since $y_1', \cdots y_1^{(n-1)}$ are integrals of (1), the equation

$$(3) \qquad B_1(y) = \sum_{k=0}^{n-1} a_k' y^{(k)} = 0$$

has the integrals $y_1, y_1', \cdots y_1^{(n-2)}$. Similarly

$$B_1'(y) = \sum_{0}^{n-1} a_k' y^{(k+1)} + \sum a_k'' y^{(k)} = 0$$

has the same integrals as (3) and hence

$$B_2(y) = \sum_{0}^{n-1} a_k'' y^{(k)} = 0$$

has the integrals $y_1, y_1', \cdots y_1^{(n-3)}$.

Continuing in this way we have finally

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* They may be also singled-valued functions of a real variable in an interval I, possessing differential coefficients of all orders up to $n - 1$ inclusive at each point of I.

$$B_{n-1}(y) = \Sigma a_k^{(n-1)} y^{(k)} = 0,$$

which has the integral y_1.

Setting $A(y) = B_0(y)$ we have the system of n equations

$$B_h(y) = 0 \qquad (h = 0, 1, 2 \cdots n - 1), \tag{4}$$

which have the common integral y_1.

The converse theorem is also true: *If the system* (4) *has a common integral* y_1, *then* $y_1, y_1', \cdots y_1^{(n-1)}$ *are integrals of* $B_0(y) = 0$; for the following is an identity

$$B_k'(y) \equiv B_k(y') + B_{k+1}(y). \tag{5}$$

By hypothesis we have

$$B_k(y_1) \equiv 0 \qquad (k = 0, 1 \cdots n - 1), \tag{6}$$

and from (5) and (6) follows immediately

$$B_k(y_1') \equiv 0 \qquad (k = 0, 1 \cdots n - 2),$$

$$B_k(y_1'') \equiv 0 \qquad (k = 0, 1 \cdots n - 3),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$B_0(y_1^{(n-1)}) \equiv 0.$$

We have thus the theorem: *The necessary and sufficient condition that the equation* (1) *have a fundamental system of the form* (2) *is that there exist an integral of the system* (4) *which is not an integral of an equation of order less than n with constant coefficients.*

II. We have shown that y_1 is an integral of the system of equations

$$\begin{aligned} y^{(n)} + \sum_0^{n-1} a_k y^{(k)} &= 0, \\ \sum_0^{n-1} a_k' y^{k)} &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \sum_0^{n-1} a_k^{(n-1)} y^{(k)} &= 0. \end{aligned} \tag{7}$$

Assume that among the $n + 1$ functions $a_0, a_1, \cdots a_{n-1}, a_n \equiv 1$, there will be a set of n which are linearly independent, *i. e.*, in the matrix

$$M = \left| \begin{array}{ccccc} 1 & a_{n-1} & \cdots & a_1 & a_0 \\ 0 & a_{n-1}' & \cdots & a_1' & a_0' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & a_{n-1}^{(n-1)} & \cdots & a_1^{(n-1)} & a_0^{(n-1)} \end{array} \right|$$

there will be one n-rowed determinant, the Wronskian of this set, which does not vanish identically. That this is always the case is shown below.

We may then solve the equations (7) in the form

$$(8)\quad y : y' : \cdots y^{(n-1)} : y^{(n)} = W_0 : - W_1 : \cdots (-1)^{n-1}W_{n-1} : (-1)^n W_n,^*$$

W_i being the Wronskian of $a_0 \cdots a_{i-1}, a_{i+1} \cdots a_n$.

Now one of the W's is not identically zero, moreover it must be one of the set $W_0 \cdots W_{n-1}$, for because of the constitution of the first column in M, these W's are the first minors of W_n which must vanish with them.[†]

Suppose W_i is the non-vanishing W and suppose further that W_j does vanish identically (j being one of the numbers $0, 1, \cdots, n-1$). We then have $y^{(i)} : y^{(j)} = W_i : W_j$ or $y^{(j)} = 0$. But we have shown that y may not satisfy an equation of order less than n with constant coefficients. Therefore W_j may not vanish identically.

We have then at once from (8)

$$(9)\quad \begin{aligned} y' &= -\frac{W_1}{W_0}y = -\alpha y, \\ y_1 &= e^{-\int \alpha dx}, \end{aligned}$$

and the equation is solved, the other integrals being obtained by differentiation.

III. Suppose every set of $k + 1$ of the functions $a_0, \cdots, a_n$ is linearly dependent but that at least one set of k functions is independent. The matrix M is then of rank k.

Consider the first k equations of (7)

$$(10)\quad \begin{aligned} y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y &= 0, \\ a_{n-1}'y^{(n-1)} + \cdots + a_0'y &= 0, \\ \cdots\cdots\cdots\cdots & \\ a_{n-1}^{(k-1)}y^{(n-1)} + \cdots + a_0^{(k-1)}y &= 0. \end{aligned}$$

If the $k - 1$ functions $a_n \equiv 1$, $a_{n-1} \cdots a_{n-k+2}$ are linearly dependent, we will add to them such other a's as will form with them a set of $k - 1$ independent functions. This set will form a fundamental system of integrals of a homogeneous linear equation of order k :

$$(11)\quad Kz^{(k-1)} + \cdots + Bz' + Az = 0.$$

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[*] Bôcher, Introduction to Higher Algebra, § 17, Theorem 4.

[†] This means that $a \ = 1$ must be included in the set, or in one of the sets, of n independent functions.

There will be an a, call it $\bar{a}$, which is not a solution of (11); otherwise there could be no linearly independent set of k a's. On the other hand every set of $k+1$ a's is linearly dependent and hence every a may be expressed linearly in terms of $\bar{a}$ and of the $k-1$ a's which form the fundamental integrals of (11), *i. e.*, every a has the form $a_k = c_k\bar{a} + \alpha_k$ (c_k being constant) where α_k is a solution of (11).

We substitute these forms for the coefficients in equations (10), multiply the equations in order by A, B, $\cdots K$, add, and, remembering that the α's are solutions of (11), we obtain:

$$(K\bar{a}^{(k-1)} + \cdots + B\bar{a}' + A\bar{a})(c_{n-1}y^{(n-1)} + \cdots + c_1y' + c_0y) = 0,$$

whence

$$c_{n-1}y^{(n-1)} + \cdots + c_1y' + c_0y = 0.$$

Not all these c's are zero, otherwise every a would be an integral of (11). y then is an integral of an equation of order less than n with constant coefficients. We showed, however, in § 1 that this was impossible, and hence our assumption that M is of rank $k<n$ leads to a contradiction.

IV. We have shown that the equation (1) cannot have integrals of the form (2) unless there is a set of n a's which are linearly independent. We have shown that in case this is true the integral is

$$y_1 = e^{-\int \frac{w_1}{w_0}dx}.$$

We may now obtain conditions which with the above will form a set of conditions necessary and sufficient that the equations $-B_k(y) \doteq 0$ have a common integral y, in case $y, y', \cdots, y^{(n-1)}$ are linearly independent. We need merely to express the fact that $y_1 = e^{-\int \frac{w_1}{w_0}dx}$ satisfies these equations. The conditions will consist of the vanishing of $n-1$ differential expressions in the a's.*

For an equation of second order

$$y'' + a_1y' + a_0y = 0$$

we have the integral

$$y_1 = e^{-\int \frac{a_0'}{a_1'}dx}$$

and the single condition †

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* It is obvious that for the equation for which $a_k = c_kx^k$, c^k being constant, these must be the conditions that the equation

$$\sum_1^n [c_km(m-1)\cdots(m-k+1)] + c_0 = 0, \quad (c_n = 1).$$

has n roots forming an arithmetical progression with common difference 1.

† For the same condition obtained somewhat differently cf. Craig, American Journal of Mathematics, No. 8.

$$(13) \qquad (a_0')^2 - a_0''a_1' + a_0'a_1'' - a_0'a_1a_1' + a_0(a_1')^2 = 0$$

For an equation of third order

$$y''' + a_2y'' + a_1y' + a_0y = 0$$

we have the integral

$$y_1 = e^{-\int \frac{a_0''a_2' - a_0'a_2''}{a_1''a_2' - a_1'a_2''}dx} = e^{-\int \alpha dx}$$

and the two conditions

$$3\alpha\alpha' - \alpha^3 - \alpha'' + a_2(\alpha^2 - \alpha') - a_1\alpha + a_0 = 0,$$

and

$$a_2'(\alpha^2 - \alpha') - a_1'\alpha + a_0' = 0,$$

and so on.

The general linear homogeneous equation of second order can always be transformed into an equation of the sort just treated. Let the equation be

$$y'' + p_1y' + p_0y = 0.$$

This goes over by the transformation

$$z = \int \frac{dx}{\varphi(x)}$$

into

$$\frac{d^2y}{dz^2} + (p_1\varphi - \varphi')\frac{dy}{dz} + p_0\varphi^2y = 0.$$

Substituting $a_1 = p_1\varphi - \varphi'$, $a_0 = p_1\varphi^2$ in the equation (13), we obtain a non-linear equation of third order for φ

If φ_1 is an integral of the equation

$$(14) \qquad \varphi' = m + p_1\varphi + \frac{p_0}{m}\varphi^2,$$

in other words, if $a_0 = -(m^2 + a_1m)$, m being a constant, the transformed equation

$$\frac{d^2y}{dz^2} + a_1\frac{dy}{dz} - (m^2 + a_1m)y = 0$$

will have an integral satisfying an equation of first order with constant coefficients, viz.: e^{mz}. Conversely, if the transformed equation has an integral of this sort, we must have $a_0 = -(m^2 + a_1m)$ or φ must be an integral of (13).

If, however, φ is any integral of (13) independent of φ_1, the transformed equation will not have an integral e^{mz} but will be of the desired form.

The general equation of nth order cannot be transformed in this way, for φ must in this case be a common integral of $n - 1$ differential equations whose coefficients are differential expressions in the coefficients of the given equation.

Illustrative problem.

$$4x^6y'' + 4x^3(2x^2 + 1)y' + (1 - 2x^2)y = 0.$$

Let $z = \int \frac{dx}{\varphi}$ and the equation becomes

$$\frac{d^2y}{dz^2} + \left[\frac{2x^2 + 1}{x^3}\varphi - \varphi'\right]\frac{dy}{dz} + \varphi^2\frac{1 - 2x^2}{4x^6}y = 0.$$

A solution for φ is seen to be $\varphi = 2x^2$ and as this is not an integral of

$$\varphi' = m + \frac{2x^2 + 1}{x^3}\varphi + \frac{1 - 2x^2}{4mx^6}\varphi^2,$$

the transformed equation

$$\frac{d^2y}{dz^2} - 4z\frac{dy}{dz} + (4z^2 - 2)y = 0$$

is of the type considered. A fundamental system of integrals is

$$y_1 = e^{-\int \frac{a_0'}{a_1'}dz} = e^{\int 2z\,dz} = e^{z^2} \quad \text{and} \quad y_1' = 2ze^{z^2}.$$

The integrals of the given equation are therefore

$$e^{\frac{1}{4x^2}} \quad \text{and} \quad \frac{1}{x}e^{\frac{1}{4x^2}}.$$

The adjoint equation is, of course, also soluble. It is

$$4x^6z'' - 4x^3(2x^2 + 1)z' + (1 + 10x^2 + 8x^4)z = 0$$

with integrals:

$$z_1 = xe^{-\frac{1}{4x^2}} \quad \text{and} \quad z_2 = -x^2e^{-\frac{1}{4x^2}}.$$

The adjoint of the transformed equation is:

$$\frac{d^2\zeta}{dz^2} + 4z\frac{d\zeta}{dz} + (4z^2 + 2)\zeta = 0,$$

with integrals:

$$\zeta_1 = 2ze^{-z^2} \quad \text{and} \quad \zeta_2 = \frac{1}{z}e^{-z^2}.$$

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